Three-manifolds with positive Ricci curvature

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Abstract

This note is a short summary of Hamilton's paper *Three-manifolds with positive Ricci curvature*(cf [1]). It will explain basic properties of Ricci flow equation and how these can be applied to geometry problems. I read this paper with Yue Wu and Yunyang Xiao, during the summer research program of The Hong Kong University of Science and Technology, advised by Prof. Frederick Fong.

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1 Evolution equations

The evolution equation we are going to study is the following:

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$$

where g_{ij} is the metric tensor and R_{ij} is the Ricci curvature tensor. Originally it is obtained by considering the Euler-Lagrange equation of total scalar curvature, i.e. integral of scalar curvature over the whole manifold. We shall see in the below that this equation has good behaviour for manifolds with positive curvature. By studying its convergence behaviour, Hamilton obtained the following result:

Theorem 1.1. *Let X be a compact 3-manifold which admits a Riemannian metric with positive Ricci curvature. Then X also admits a metric of constant positive curvature.*

Precisely, we are going to show that in dimension three, the Ricci flow equation admits a unique smooth solution converging to a Einstein metric at infinity. The main theorem follows from the speciality of dimension. As a corollary, any simply-connected threefold with a positive Ricci curvature metric must be diffeomorphic to S^3.

To begin with, we need to show the existence of solutions for short time. This is first proved by Hamilton:

Theorem 1.2. *The evolution equation $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$ has a solution for short time on any closed manifold X^n with any initial Riemannian metric.*

We won't prove this theorem here. The orginal proof of Hamilton is rather complicated since the linearization of equation is not strictly parabolic. Later in 1983, DeTurck gave a simple proof of short existence(cf [2]). He modified the equation by a diffeomorphism, so that the new equation is strictly parabolic. In this case, the usual existence theorem can be applied.

The degeneracy is due to the full symmetry of the equation. In fact, the diffeomorphism group of X acts on the space of Riemannian metrics by pull-back and the equation is invariant under the this action. So the desired ellipticity shall be obtained only after modulo diffeomorphims. This is the trick that DeTurck used in his paper. And the phenomenon is rather similar to the case of Yang-Mills equations(cf [3][4]), in which case we should exploit Uhlenbeck's gauge fixing theorem to deduce ellipticity.

2 Tensor maximum principle

The most powerful tool in the study of evolution equations is the maximum principle. To apply this tool to our equation, we need to generalize it for tensors, which was formulated by Hamilton in this paper.

Definition 2.1. *Given a map $p : Sym^2T^*M \to Sym^2T^*M$, we say it satisfies the* **null-eigenvector assumption** *if for any symmetric $(0,2)$-tensor M_{ij} and its null-vector v^i, we have $p(M_{ij})(v,v) \geq 0$. Here null-vector means $M_{ij}v^i = 0$ for any j.*

Let $g_{ij}(t)$ and $M_{ij}(t)$ be smooth one-parameter family of metrics and symmetric $(0,2)$-tensors. Let $N_{ij} = p(M_{ij}, g_{ij})$ be a polynomial formed by contracting products of M_{ij} via the metric g_{ij}. Under such setting, Hamilton proved:

Theorem 2.2 (Tensor Maximum Principle)**.** *Suppose that on $0 \leq t \leq T$ we have*

$$\frac{\partial}{\partial t} M_{ij} \geq \Delta M_{ij} + u^k M_{ij,k} + N_{ij}$$

where u^k is some smooth vector field, possibly time-dependent. Assume $N_{ij} = p(M_{ij}, g_{ij})$ satisfies the null-eigenvector condition, then if $M_{ij} \geq 0$ at $t = 0$, then it remains so on $0 \leq t \leq T$.

Note that this has nothing to with the dimension and the family of metrics need not to satisfy the Ricci flow equation. This phenomenon will be frequent in the following. We will transfer between general setting and three-dimension Ricci flow case.

Proof. We proof by continuity method. It suffices to show that if $M_{ij} \geq 0$ at $t = 0$, then it remains so on $0 \leq t \leq \delta$ for some $\delta > 0$.

Consider the tensor

$$\tilde{M}_{ij} = M_{ij} + \epsilon(\delta + t) g_{ij}$$

where δ is to be selected. We claim that $\tilde{M}_{ij} > 0$ on $0 \leq t \leq \delta$ for every $\epsilon > 0$. Then let $\epsilon \to 0$ will finish the proof.

Let θ =sup$\{t \in [0, \delta] : \tilde{M}_{ij} > 0$ on $[0, t]\}$. Since $\tilde{M}_{ij}(0) = M_{ij}(0) + \epsilon g_{ij}(0)$, we must have $\tilde{M}_{ij}(0) > 0$. Then continuity shows that $\theta > 0$.

If our claim fails, then there exists a null-eigenvector v^i of unit length at some point $x \in X$. Let $\tilde{N}_{ij} = p(\tilde{M}_{ij}, g_{ij})$, then by assumption we have $\tilde{N}_{ij}v^iv^j \geq 0$ at (x, θ).

Since p is a polynomial, there exists constant $C_1 > 0$ depending only on sup$|M_{ij}|$, such that

$$|\tilde{N}_{ij} - N_{ij}| \leq C_1 |\tilde{M}_{ij} - M_{ij}|$$

Therefore we have

$$|\tilde{N}_{ij}v^iv^j - N_{ij}v^iv^j| \leq C_1 |\epsilon(\theta + \delta) g_{ij} v^i v^j| \leq C_1 \epsilon \delta$$

Hence

$$N_{ij}v^iv^j \geq \tilde{N}_{ij}v^iv^j - C_1\epsilon\delta \geq -C_1\epsilon\delta$$

Here C_1 depends only on sup$|M_{ij}|$ and g_{ij}, but not on ϵ and δ.

We extend v^i to a vector field in a neighborhood of x with $v^i_{,j} = 0$ at x. For example, we can parallel transport v^i along radial geodesics. This extended vector field is independent of t. Let $f = \tilde{M}_{ij}v^iv^j$, then $f \geq 0$ on $0 \leq t \leq \theta$

and all of X. Since $f = 0$ at (x, θ), we see that $\frac{\partial f}{\partial t} \leq 0$, $f_{,k} = 0$ and $\Delta f \geq 0$ at (x, θ).

Now choose $C_2 > 0$ sufficiently large such that

$$\frac{\partial}{\partial t} g_{ij} \geq -C_2 g_{ij}$$

for all $t \in [0, T]$. Then at (x, θ) we have

$$0 \geq \frac{\partial f}{\partial t} = \frac{\partial M_{ij}}{\partial t} v^i v^j + \epsilon + \epsilon(\theta + \delta) \frac{\partial g_{ij}}{\partial t} v^i v^j \geq \frac{\partial M_{ij}}{\partial t} v^i v^j + (1 - 2\delta C_2)\epsilon$$

Since $v^i_{,j} = 0$ and $\tilde{M}_{ij} v^i = 0$, we have

$$f_{,k} = M_{ij,k} v^i v^j, \Delta f = \Delta M_{ij} v^i v^j$$

Together with the parabolic inequality, we see at (x, θ)

$$\frac{\partial}{\partial t} M_{ij} v^i v^j \geq \Delta M_{ij} v^i v^j + u^k M_{ij,k} v^i v^j + N_{ij} v^i v^j \geq N_{ij} v^i v^j$$

Combining the above inequalities, we have

$$C_1 \delta \geq 1 - 2C_2 \delta$$

This gives contradiction if $\delta < (C_1 + 2C_2)^{-1}$. □

The following examples illustrate how this works. Suppose now that $g_{ij}(t)$ is a smooth solution of Ricci flow equation in dimension three defined on $[0, T)$, where T is the maximal existence time. Then we have:

Theorem 2.3. *If $Ric \geq \epsilon R g > 0$ at $t = 0$, then it remains so on $[0, T)$, where ϵ is some positive constant and R is the scalar curvature.*

Proof. Consider the symmetric tensor $M_{ij} = \frac{R_{ij}}{R} - \epsilon g_{ij}$. Direct computation shows that:

$$\frac{\partial}{\partial t} M_{ij} = \Delta M_{ij} + (\frac{2}{R} g^{kl} R_{,l}) M_{ij,k} + (2\epsilon R_{ij} - \frac{R Q_{ij} + 2 S R_{ij}}{R^2})$$

where Q_{ij} and S are some tensors obtained from curvature tensor. By diagonalize the tensor at one point, we can show that the final term satisfies the null-eigenvector condition. By tensor maximum principle, M_{ij} remains semi-positive for all time. □

Similarly we can prove the following results:

(i) The eigenvalues of Ricci tensor approach each other at every point;

(ii) Estimate of $|\nabla R|$ in terms of R, so in particular we can obtain $R_{max}/R_{min} \to 1$ as $t \to T$.

3 Long time existence

Here we explain how to obtain long time existence of Ricci flow equation. In fact, Ricci flow equation of the form $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$ admits no solution of infinite time. Precisely, we have:

lemma 3.1. *If $R \geq \rho > 0$ at $t = 0$, then $T \leq \frac{3}{2\rho}$.*

Proof. Direct computation shows that:

$$\frac{\partial}{\partial t} R \geq \Delta R + \frac{2}{3} R^2$$

Consider the ODE:

$$\frac{df}{dt} = \frac{2}{3} f^2$$

with $f = \rho$ at $t = 0$. Explicitly the solution is given by:

$$f = \frac{3\rho}{3 - 2\rho t}$$

Taking f as a function on $X \times [0, \frac{3}{2\rho})$ constant on X direction. Then we have:

$$\frac{\partial}{\partial t}(R - f) \geq \Delta(R - f) + \frac{2}{3}(R + f)(R - f)$$

And maximum principle implies that $R - f \geq 0$ on $0 \leq t < T$. Since $f \to \infty$ as $t \to \frac{3}{2\rho}$, we must have $T \leq \frac{3}{2\rho}$. □

So what we mean by long time existence is not for this version of equation. We will consider the normalized version of Ricci flow equation as follows.

Let $\tilde{g}_{ij}(t) = \psi(t) g_{ij}(t)$ such that:

$$\int_X d\tilde{\mu}(t) \equiv 1$$

That is, the volume is preserved under flow. This explains the name "normalized equation". Also, we set $\tilde{t} = \int \psi(t) dt$. Under such transformation, the flow equation now becomes:

$$\frac{\partial}{\partial t} \tilde{g}_{ij} = \frac{2}{3} \tilde{r} \tilde{g}_{ij} - 2\tilde{R}_{ij}$$

where $\tilde{r}$ is the average of scalar curvature. In this case, we have:

Theorem 3.2. *The solution exists for all time, i.e. $\tilde{T} = \infty$.*

To prove this, we need several steps. The first and hardest step is to show that for unnormalized equation, $R_{max} \to \infty$ as $t \to T$.

To prove this, we need the following general result(i.e., this holds in any dimension):

Theorem 3.3. *Suppose the evolution equation* $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$ *has a unique solution on a maximal time interval* $[0, T)$. *If* $T < \infty$, *then* $sup_X |R_{ijkl}| \to \infty$ *as* $t \to T$.

We sketch the proof of this theorem. Technically, we need a condition which assures the convergence of metrics. The following lemma is the core of argument:

lemma 3.4. *Let* $g_{ij}(t)$ *be a one-parameter family of smooth metrics on* X *for* $0 \le t < T$ *(where* T *is not necessarily finite). Suppose:*

$$\int_0^T sup_X |g'_{ij}(t)| dt < \infty$$

Then the metrics for different times are uniformly equivalent, and they C^0*-converge to a continuous positive-definite symmetric tensor* $g_{ij}(T)$.

So suppose T is finite and curvature is bounded, then this integral is bounded and the metrics C^0-converge to a continuous metric. *A priori* estimates of higher derivatives then give rise to the C^∞-convergence. Apply the short existence result, this contradicts with the maximality of T.

In our case, we have shown that T is finite, so $R_{max} \to \infty$ as $t \to T$ due to the positity of Ricci curvature. As a corollary, we see:

Corollary 3.5. *We have:*

$$\int_0^T R_{max} dt = \infty$$

Proof. Consider the following ODE:

$$\frac{df}{dt} = 2R_{max} f$$

with $f(0) = R_{max}(0)$. By direct computation we can show:

$$\frac{\partial}{\partial t}(R - f) \le \Delta(R - f) + 2R_{max}(R - f)$$

Hence $R \le f$ on $0 \le t < T$ by maximum principle. Since $R_{max} \to \infty$ as $t \to T$, we have $f \to \infty$. However, by definition of f we have:

$$log f(t)/f(0) = 2 \int_0^t R_{max}(\theta) d\theta$$

Therefore the integral diverges as $t \to T$. □

Corollary 3.6. *We have:*

$$\int_0^T r dt = \infty$$

Proof. The result follows from previous corollary and two facts: $R_{min} \le r \le R_{max}$ and $R_{max}/R_{min} \to 1$ as $t \to T$. □

Our final lemmas are the following:

lemma 3.7. *We have:*

(i) $\tilde{R}_{max}/\tilde{R}_{min} \to 1$ *as* $\tilde{t} \to \tilde{T}$*;*

(ii) $\tilde{R}_{ij} \geq \epsilon \tilde{R}\tilde{g}_{ij}$ *for some* $\epsilon > 0$*;*

(iii) $\tilde{R}_{max} \leq C < \infty$ *on* $0 \leq \tilde{t} < \tilde{T}$.

Proof. The first two follow from considering the scaling factor and what we have proved for unnormalized equation. It suffices to prove the third assertion.

Since $\tilde{R}_{ij} \geq \epsilon \tilde{R}\tilde{g}_{ij} > 0$, by volume comparison theorem we have:

$$\tilde{V} \leq C\tilde{d}^3$$

By Myers' theorem, $\tilde{d} \leq C\tilde{R}_{min}^{-\frac{1}{2}}$. Thus we have $\tilde{V}\tilde{R}_{min}^{\frac{3}{2}} \leq C$. But $\tilde{V} \equiv 1$ for normalized equation, so we see that $\tilde{R}_{min}$ is bounded. Combined with first assertion, we see that $\tilde{R}_{max}$ is bounded. □

Now we are in the position to show the solution exists for all time. By definition of normalized equation, we have:

$$\int_0^{\tilde{T}} \tilde{r} d\tilde{t} = \int_0^T r dt = \infty$$

However $\tilde{r} \leq \tilde{R}_{max} \leq C$, so $\tilde{T}$ must be ∞.

4 Convergence of Ricci flow

Now the solution of normalized Ricci flow equation exists for all time. Still, we want to use Lemma 3.4 to deduce the convergence of metrics. But for normalized equation, maximal time $\tilde{T}$ is not finite, and the right side of equation has an extra term $\frac{2}{3}\tilde{r}\tilde{g}_{ij}$. So we should make suitable decaying estimate for the right side terms. This is the following:

Theorem 4.1. *We have exponential decaying estimate:*

$$|\tilde{R}_{ij} - \frac{1}{3}\tilde{r}\tilde{g}_{ij}| \leq Ce^{-\delta\tilde{t}}$$

Corollary 4.2. *The metrics* $\tilde{g}_{ij}(t)$ *are uniformly equivalent, and* C^0*-converge to a continuous Riemannian metric* $\tilde{g}_{ij}(\infty)$*.*

We explain where the exponential term comes from, but omit the detailed calculation. From this general observation it's easy to obtain exponential decaying estimates for all relevent terms.

Suppose we have some equations in unnormalized case:

$$\frac{\partial P}{\partial t} = \Delta P + Q$$

Then normalization will give:

$$\frac{\partial \tilde{P}}{\partial \tilde{t}} = \tilde{\Delta}\tilde{P} + \tilde{Q} + \frac{2}{3}n\tilde{r}\tilde{P}$$

where n is some integer coming from scaling process. Therefore a suitable choice of constant will give rise to:

$$\frac{\partial}{\partial t}(e^{\delta\tilde{t}}\tilde{P}) \leq \tilde{\Delta}(e^{\delta\tilde{t}}\tilde{P}) + \tilde{Q}'$$

The maximum principle will then imply the exponential convergence.

Combining the above technique and Sobolev inequalities, we can similarly obtain estimates of higher derivatives:

Theorem 4.3. *For every $k > 0$, we have:*

$$sup_X|\nabla^k \tilde{R}c| \leq Ce^{-\delta\tilde{t}}$$

Corollary 4.4. *The metrics $g_{ij}(t)$ C^∞-converge to a smooth Riemannian metric $\tilde{g}_{ij}(\infty)$. And the limit metric has constant positive curvature.*

Proof. Since in dimension three, the Riemannian curvature can be recovered from Ricci curvature, thus we obtain uniform C^k-estimate of g_{ij} for all k. Therefore the metrics converge smoothly to a smooth metric. The limit is Einstein due to Theorem 4.1, and has constant curvature since we are in dimension three. □

Acknowledgements

I want to thank for Prof. Fong's guidance when we are reading Hamilton's paper. Some specific details were pointed out to us by Prof. Fong, such as a minor mistake of Hamilton's proof of tensor maximum principle and some historical remarks on further development of Ricci flows. Also I'm happy to learn this paper together with Yue Wu and Yunyang Xiao. I think I learned a lot from your presentations.

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